Conventional Accelerator Magnet Design

Neil Marks,
STFC- ASTeC/Cockcroft Institute – U. of Liverpool,
Daresbury Laboratory,
Warrington WA4 4AD,
U.K.
Tel: (44) (0)1925 603191
Fax: (44) (0)1925 603192
n.marks@dl.ac.uk
1. Present a overview of electro- magnetic technology as used in particle accelerators, considering only warm, d.c. magnets (ie superconducting magnets and a.c. effects excluded).

2. Provide an overview of the use of finite element codes for practical magnet designs, focusing specifically on OPERA 2D (Vector Fields, Kiddlington) to support a practical exercise on the two-dimensional design of a dipole (and possibly quadrupole) magnet using parameters for the accelerator and compressor rings for a neutrino factory proton driver.
Theory
• Maxwell's 2 magneto-static equations;

With no currents or steel present:
• Solutions in two dimensions with scalar potential (no currents);
• Cylindrical harmonic in two dimensions (trigonometric formulation);
• Field lines and potential for dipole, quadrupole, sextupole;

Introduction of steel:
• Ideal pole shapes for dipole, quad and sextupole;
• Field harmonics-symmetry constraints and significance;
• Significance and use of contours of constant vector potential;

Three dimensional issues:
• Termination of magnet ends and pole sides;
• The ‘Rogowski roll-off’
Introduction of currents:
- Ampere-turns in dipole, quad and sextupole;
- Coil design;
- Coil economic optimisation-capital/running costs;
- The magnetic circuit-steel requirements-permeability and coercivity;

Practical Issues:
- Backleg and coil geometry- 'C', 'H' and 'window frame' designs;
- FEA techniques - Modern codes- OPERA 2D; TOSCA.
- Judgement of magnet suitability in design.
- Demonstration of OPERA 2D

Possible end-game:
- Complex formulation.
Maxwell’s equations: \( \nabla \cdot \mathbf{B} = 0 \); 
\( \nabla \times \mathbf{H} = \mathbf{j} \);

Then we can put: \( \mathbf{B} = -\nabla \phi \)

So that: \( \nabla^2 \phi = 0 \) (Laplace's equation).

Taking the two dimensional case (ie constant in the \( z \) direction) and solving for cylindrical coordinates \( (r, \theta) \):

\[
\phi = (E + F \theta)(G + H \ln r) + \sum_{n=1}^{\infty} \left( J_n r^n \cos n\theta + K_n r^n \sin n\theta + L_n r^{-n} \cos n\theta + M_n r^{-n} \sin n\theta \right)
\]
In practical situations:

The scalar potential simplifies to:

$$\phi = \sum_n (J_n r^n \cos n\theta + K_n r^n \sin n\theta),$$

with $n$ integral and $J_n, K_n$ a function of geometry.

Giving components of flux density:

$$B_r = - \sum_n (n J_n r^{n-1} \cos n\theta + nK_n r^{n-1} \sin n\theta)$$
$$B_\theta = - \sum_n (-n J_n r^{n-1} \sin n\theta + nK_n r^{n-1} \cos n\theta)$$
This is an infinite series of cylindrical harmonics; they define the allowed distributions of $B$ in 2 dimensions in the absence of currents within the domain of $(r,\theta)$.

Distributions not given by above are not physically realisable.

Coefficients $J_n$, $K_n$ are determined by geometry (remote iron boundaries and current sources).

Note that this formulation can be expressed in terms of complex fields and potentials.
To obtain these equations in Cartesian coordinates, expand the equations for $\phi$ and differentiate to obtain flux densities;

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta; \quad \cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta;
\]
\[
\sin 2\theta = 2 \sin \theta \cos \theta; \quad \sin 3\theta = 3\sin^2 \theta - \sin^3 \theta;
\]
\[
\cos 4\theta = \cos^4 \theta + \sin^4 \theta - 6 \cos^2 \theta \sin^2 \theta;
\]
\[
\sin 4\theta = 4 \sin \theta \cos^3 \theta - 4 \sin^3 \theta \cos \theta;
\]

etc (messy!);

\[
x = r \cos \theta; \quad y = r \sin \theta;
\]

and

\[
B_x = - \frac{\partial \phi}{\partial x}; \quad B_y = - \frac{\partial \phi}{\partial y}
\]
Conventional Accelerator Magnets

Cylindrical:
\[ \phi = J_1 r \cos \theta + K_1 r \sin \theta. \]
\[ B_r = J_1 \cos \theta + K_1 \sin \theta; \]
\[ B_\theta = -J_1 \sin \theta + K_1 \cos \theta; \]

Cartesian:
\[ \phi = J_1 x + K_1 y \]
\[ B_x = -J_1 \]
\[ B_y = -K_1 \]

So, \( J_1 = 0 \) gives vertical dipole field:
\( K_1 = 0 \) gives horizontal dipole field.
Quadrupole field n=2:

Cylindrical:
\[ \phi = J_2 r^2 \cos 2\theta + K_2 r^2 \sin 2\theta; \]
\[ B_r = 2 J_2 r \cos 2\theta + 2K_2 r \sin 2\theta; \]
\[ B_\theta = -2J_2 r \sin 2\theta + 2K_2 r \cos 2\theta; \]

Cartesian:
\[ \phi = J_2 (x^2 - y^2) + 2K_2 xy \]
\[ B_x = -2 (J_2 x + K_2 y) \]
\[ B_y = -2 (-J_2 y + K_2 x) \]

\( J_2 = 0 \) gives 'normal' or 'upright' quadrupole field.

\( K_2 = 0 \) gives 'skew' quad fields (above rotated by \( \pi/4 \)).
Sextupole field $n=3$:

**Cylindrical:**
\[
\phi = J_3 \ r^3 \cos 3\theta + K_3 \ r^3 \sin 3\theta; \\
B_r = 3 \ J_3 r^2 \ \cos 3\theta + 3K_3 r^2 \ \sin 3\theta; \\
B_\theta = -3J_3 \ r^2 \ \sin 3\theta + 3K_3 \ r^2 \ \cos 3\theta;
\]

**Cartesian:**
\[
\phi = J_3 \ (x^3-3y^2x)+K_3(3yx^2-y^3) \\
B_x = -3\{J_3 \ (x^2-y^2)+2K_3 yx\} \\
B_y = -3\{-2J_3 \ xy +K_3(x^2-y^2)\}
\]

$J_3 = 0$ giving 'normal' or 'right' sextupole field.
Summary: variation of \( B_y \) on \( x \) axis.

Dipole; constant field:

Quad; linear variation:

Sextupole: quadratic variation:
Alternative notation:

\[ B(x) = B \rho \sum_{n=0}^{\infty} \frac{k_n x^n}{n!} \]

magnet strengths are specified by the value of \( k_n \); (normalised to the beam rigidity);

order \( n \) of \( k \) is different to the 'standard' notation:

- dipole is \( n = 0 \);
- quad is \( n = 1 \); etc.

\( k \) has units:

- \( k_0 \) (dipole) \( m^{-1} \);
- \( k_1 \) (quadrupole) \( m^{-2} \); etc.
Introducing iron yokes and poles.

What is the ideal pole shape?

• Flux is normal to a ferromagnetic surface with infinite $\mu$:
  
  $\mu = \infty$

  $\mu = 1$

  $\text{curl } H = 0$

  therefore $\int H \cdot ds = 0$

  in steel $H = 0$

  therefore parallel $H$ air $= 0$

  therefore $B$ is normal to surface.

• Flux is normal to lines of scalar potential, ($\mathbf{B} = -\nabla \phi$);

• So the lines of scalar potential are the ideal pole shapes!

(but these are infinitely long!)
Equations of ideal poles

Equations for Ideal (infinite) poles;
\( J_n = 0 \) for normal (ie not skew) fields:

**Dipole:**
\[ y = \pm \frac{g}{2}; \]
\((g\) is inter-pole gap).

**Quadrupole:**
\[ xy = \pm \frac{R^2}{2}; \]

**Sextupole:**
\[ 3x^2y - y^3 = \pm R^3; \]
'Combined Function Magnets' - often dipole and quadrupole field combined (but see next-but-two slide):

A quadrupole magnet with physical centre shifted from magnetic centre.

Characterised by 'field index' $n$, +ve or -ve depending on direction of gradient; do not confuse with harmonic $n$!

$$n = -\frac{\rho}{B_0} \left( \frac{\partial B}{\partial x} \right),$$

$\rho$ is radius of curvature of the beam;

$B_0$ is central dipole field.
Combined function (dipole & quadrupole) magnet:

- beam is at physical centre
- flux density at beam \( = B_0 \)
- gradient at beam \( = \frac{\partial B}{\partial x} \)
- magnetic centre is at \( B = 0 \)
- separation magnetic to physical centre \( = X_0 \)

magnetic centre, \( x' = 0 \)

physical centre \( x = 0 \)
Pole for a combined dipole and quadrupole.

Physical and magnetic centres are separated by $X_0$.

Horizontal displacement from true quad centre is $x'$.

Then

$$B_0 = \left( \frac{\partial B}{\partial x} \right) X_0$$

therefore

$$x' y = \pm \frac{R^2}{2}$$

As

$$x' = x + X_0$$

Pole equation is

$$y = \pm \frac{R^2}{2} \frac{n}{\rho} \left( 1 - \frac{n x}{\rho} \right)^{-1}$$

or

$$y = \pm g \left( 1 - \frac{n x}{\rho} \right)^{-1}$$

where $g$ is the half gap at the physical centre of the magnet.

Rewritten as

$$y = \pm g \left[ 1 - \frac{x}{B_0} \left( \frac{\partial B}{\partial x} \right) \right]^{-1}$$
Other combined function magnets

Other combinations:

• dipole, quadrupole and sextupole;
• dipole & sextupole (for chromaticity control);
• dipole, skew quad, sextupole, octupole (at DL)

Generated by

• pole shapes given by sum of correct scalar potentials
  - amplitudes built into pole geometry – not variable.
• multiple coils mounted on the yoke
  - amplitudes independently varied by coil currents.
Practically, poles are finite, **introducing errors**; these appear as higher harmonics which degrade the field distribution.

However, the iron geometries have certain symmetries that **restrict** the nature of these errors.

**Dipole:**

**Quadrupole:**
Possible symmetries.

Lines of symmetry:

Dipole: 
\[ y = 0; \]

Quad 
\[ x = 0; \quad y = 0 \]

Pole orientation determines whether pole is normal or skew.

Additional symmetry imposed by pole edges:

\[ x = 0; \quad y = \pm x \]

The additional constraints imposed by the symmetrical pole edges limits the values of \( n \) that have non-zero coefficients.
### Dipole symmetries

<table>
<thead>
<tr>
<th>Type</th>
<th>Symmetry</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole orientation</td>
<td>( \phi(\theta) = -\phi(-\theta) )</td>
<td>all ( J_n = 0 );</td>
</tr>
<tr>
<td>Pole edges</td>
<td>( \phi(\theta) = \phi(\pi - \theta) )</td>
<td>( K_n ) non-zero only for:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( n = 1, 3, 5, \text{ etc}; )</td>
</tr>
</tbody>
</table>

So, for a fully symmetric dipole, only 6, 10, 14 etc pole errors can be present.
## Quadrupole symmetries

<table>
<thead>
<tr>
<th>Type</th>
<th>Symmetry</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole orientation</td>
<td>( \phi(\theta) = -\phi(-\theta) )</td>
<td>All ( J_n = 0; )</td>
</tr>
<tr>
<td></td>
<td>( \phi(\theta) = -\phi(\pi - \theta) )</td>
<td>( K_n = 0 ) all odd ( n; )</td>
</tr>
<tr>
<td>Pole edges</td>
<td>( \phi(\theta) = \phi(\pi/2 - \theta) )</td>
<td>( K_n ) non-zero</td>
</tr>
<tr>
<td></td>
<td></td>
<td>only for: ( n = 2, 6, 10, ) etc;</td>
</tr>
</tbody>
</table>

So, a fully symmetric quadrupole, only 12, 20, 28 etc pole errors can be present.
### Sextupole symmetries.

<table>
<thead>
<tr>
<th>Type</th>
<th>Symmetry</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pole orientation</td>
<td>$\phi(\theta) = -\phi(-\theta)$</td>
<td>All $J_n = 0$;</td>
</tr>
<tr>
<td></td>
<td>$\phi(\theta) = -\phi(2\pi/3 - \theta)$</td>
<td>$K_n = 0$ for all $n$</td>
</tr>
<tr>
<td></td>
<td>$\phi(\theta) = -\phi(4\pi/3 - \theta)$</td>
<td>not multiples of 3;</td>
</tr>
<tr>
<td>Pole edges</td>
<td>$\phi(\theta) = \phi(\pi/3 - \theta)$</td>
<td>$K_n$ non-zero only</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for: $n = 3, 9, 15, \text{etc.}$</td>
</tr>
</tbody>
</table>

18, 30, 42, etc pole errors can be present.
Summary of ‘allowed harmonics’ in **fully symmetric** magnets:

<table>
<thead>
<tr>
<th>Fundamental geometry</th>
<th>‘Allowed’ harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole, n = 1</td>
<td>n = 3, 5, 7, ......</td>
</tr>
<tr>
<td></td>
<td>( 6 pole, 10 pole, etc.)</td>
</tr>
<tr>
<td>Quadrupole, n = 2</td>
<td>n = 6, 10, 14, ....</td>
</tr>
<tr>
<td></td>
<td>(12 pole, 20 pole, etc.)</td>
</tr>
<tr>
<td>Sextupole, n = 3</td>
<td>n = 9, 15, 21, ...</td>
</tr>
<tr>
<td></td>
<td>(18 pole, 30 pole, etc.)</td>
</tr>
<tr>
<td>Octupole, n = 4</td>
<td>n = 12, 20, 28, ....</td>
</tr>
<tr>
<td></td>
<td>(24 pole, 40 pole, etc.)</td>
</tr>
</tbody>
</table>
We have: \( \vec{B} = \text{curl} \ \vec{A} \) \hspace{1cm} (\vec{A} \text{ is vector potential});

and \( \text{div} \ \vec{A} = 0 \)

Expanding:
\[
\vec{B} = \text{curl} \ \vec{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) \mathbf{i} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \mathbf{j} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \mathbf{k} ;
\]

where \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), and unit vectors in \( x, y, z \).

In 2 dimensions \( B_z = 0; \ \frac{\partial}{\partial z} = 0; \)

So \( A_x = A_y = 0; \)

and \( \vec{B} = (\frac{\partial A_z}{\partial y}) \mathbf{i} - (\frac{\partial A_z}{\partial x}) \mathbf{j} \)

\( \vec{A} \) is in the \( z \) direction, normal to the 2 D problem.

Note: \( \text{div} \ \vec{B} = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_z}{\partial x \partial y} = 0; \)
In a two dimensional problem the magnetic flux between two points is proportional to the difference between the vector potentials at those points.

\[ \Phi \propto (A_2 - A_1) \]

Proof on next slide.
Consider a rectangular closed path, length \( \lambda \) in z direction at \((x_1, y_1)\) and \((x_2, y_2)\); apply Stokes’ theorem:

\[
\Phi = \iint B \cdot dS = \iint (\text{curl } A) \cdot dS = \oint A \cdot ds
\]

But \( A \) is exclusively in the z direction, and is constant in this direction. So:

\[
\int A \cdot ds = \lambda \{ A(x_1, y_1) - A(x_2, y_2) \};
\]

\[
\Phi = \lambda \{ A(x_1, y_1) - A(x_2, y_2) \};
\]
Therefore:

Contours of constant vector potential in 2D give a graphical representation of lines of flux.

These are used in 2D FEA analysis to obtain a graphical image of flux distribution.
Fringe flux will be present at pole ends so beam deflection continues beyond magnet end:

The magnet’s strength is given by \( \int B_y(z) \, dz \) along the magnet, the integration including the fringe field at each end;

The ‘magnetic length’ is defined as \( (1/B_0)(\int B_y(z) \, dz) \) over the same integration path, where \( B_0 \) is the field at the azimuthal centre.
At high(ish) fields it is necessary to terminate the pole (transverse OR longitudinal) in a controlled way:
• to prevent saturation in a sharp corner (see diagram);
• to maintain length constant with \( x, y \);
• to define the length (strength) or preserve quality;
• to prevent flux entering normal to lamination (ac).

Longitudinally, the end of the magnet is therefore 'chamfered' to give increasing gap (or inscribed radius) and lower fields as the end is approached.
The 'Rogowski' roll-off:

Equation: \[ y = \frac{g}{2} + \left(\frac{g}{pa}\right) [\exp (apx/g) - 1]; \]

\( g/2 \) is dipole half gap;
\( y = 0 \) is centre line of gap;
\( a \) is a parameter controlling gradient at \( x = 0 \) (~1).

This profile provides the maximum rate of increase in gap with a monotonic decrease in flux density at the surface ie no saturation.
As the gap is increased, the size (area) of the shim is increased, to give *some* control of the field quality at the lower field. This is far from perfect!

Transverse adjustment at end of dipole

Transverse adjustment at end of quadrupole
Calculation of end effects using 2D codes

FEA model in longitudinal plane, with correct end geometry (including coil), but 'idealised' return yoke:

This will establish the end distribution; a numerical integration will give the 'B' length.

Provided dB/dz is not too large, single 'slices' in the transverse plane can be used to calculated the radial distribution as the gap increases. Again, numerical integration will give \( \int B \, dl \) as a function of x.

This technique is less satisfactory with a quadrupole, but end effects are less critical with a quad.
Now for $j \neq 0$  

$\nabla \mathbf{H} = \mathbf{j}$;

To expand, use Stoke’s Theorem: for any vector $\mathbf{V}$ and a closed curve $s$:

$$\int \mathbf{V} \cdot ds = \iint \text{curl} \mathbf{V} \cdot dS$$

Apply this to: $\text{curl} \mathbf{H} = \mathbf{j}$;

then in a magnetic circuit:

$$\int \mathbf{H} \cdot ds = NI;$$

$NI$ (Ampere-turns) is total current cutting $S$
Excitation current in a dipole

B is approx constant round the loop made up of λ and g, (but see below);

But in iron, \( \mu \gg 1 \), and

\[
H_{\text{iron}} = \frac{H_{\text{air}}}{\mu};
\]

So

\[
B_{\text{air}} = \frac{\mu_0 NI}{(g + \lambda/\mu)};
\]

\( g \), and \( \lambda/\mu \) are the 'reluctance' of the gap and iron.

Approximation ignoring iron reluctance (\( \lambda/\mu << g \)):

\[
NI = B g /\mu_0
\]
For quadrupoles and sextupoles, the required excitation can be calculated by considering fields and gap at large x. For example:

**Quadrupole:**

Pole equation: \( xy = \frac{R^2}{2} \)

On x axes \( B_y = gx; \)

where \( g \) is gradient (T/m).

At large x (to give vertical lines of B):

\[
N I = (gx) \left( \frac{R^2}{2x} \right)/\mu_0
\]

ie \( N I = g \frac{R^2}{2} \mu_0 \) (per pole).

**Sextupole,**

The same method for a Sextupole, (coefficient \( g_s \)), gives:

\[
N I = g_s \frac{R^3}{3} \mu_0 \) (per pole)
In air (remote currents!),

\[ B = \mu_0 H \]
\[ \mathbf{B} = - \nabla \phi \]

Integrating over a limited path (not circular) in air:

\[ NI = (\phi_1 - \phi_2)/\mu_0 \]

\( \phi_1, \phi_2 \) are the scalar potentials at two points in air. Define \( \phi = 0 \) at magnet centre; then potential at the pole is:

\[ \mu_0 NI \]

Apply the general equations for magnetic field harmonic order \( n \) for non-skew magnets (all \( J_n = 0 \)) giving:

\[ NI = (1/n) \left( 1/\mu_0 \right) \left\{ B_r/R^{(n-1)} \right\} R^n \]

Where:

- \( NI \) is excitation per pole;
- \( R \) is the inscribed radius (or half gap in a dipole);
- term in brackets \( \{ \} \) is magnet strength in T/m \( (n-1) \).
Standard design is rectangular copper (or aluminium) conductor, with cooling water tube. Insulation is glass cloth and epoxy resin.

Amp-turns (NI) are determined, but total copper area ($A_{\text{copper}}$) and number of turns (N) are two degrees of freedom and need to be decided.

Current density: $j = \frac{NI}{A_{\text{copper}}}$

Optimum $j$ determined from **economic** criteria.
Advantages of low j:
• **lower power loss** – power bill is decreased;
• **lower power loss** – power converter size is decreased;
• **less heat** dissipated into magnet tunnel.

Advantages of high j:
• **smaller coils**;
• **lower capital cost**;
• **smaller magnets**.

Chosen value of j is an optimisation of magnet capital against power costs.
Number of turns per coil, N

The value of number of turns (N) is chosen to match power supply and interconnection impedances.

Factors determining choice of N:

**Large N (low current)**
- Small, neat terminals.
- Thin interconnections—hence low cost and flexible.
- More insulation layers in coil, hence larger coil volume and increased assembly costs.
- High voltage power supply - safety problems.

**Small N (high current)**
- Large, bulky terminals
- Thick, expensive connections.
- High percentage of copper in coil volume. More efficient use of space available
- High current power supply. - greater losses.
### Examples of typical turns/currents

From the Diamond 3 GeV synchrotron source:

**Dipole:**
- \( N \) (per magnet): 40
- \( I_{\text{max}} \) 1500 A
- Volts (circuit): 500 V

**Quadrupole:**
- \( N \) (per pole): 54
- \( I_{\text{max}} \) 200 A
- Volts (per magnet): 25 V

**Sextupole:**
- \( N \) (per pole): 48
- \( I_{\text{max}} \) 100 A
- Volts (per magnet): 25 V
Yoke - Permeability of low silicon steel

μ

B (T)

Parallel to rolling direction
Normal to rolling direction.

JAI Presentation 2010
Conventional Accelerator Magnets
N. Marks, CI / U. of Liverpool.
Flux in the yoke includes the gap flux and stray flux, which extends (approx) one gap width on either side of the gap.

Thus, to calculate total flux in the back-leg of magnet length \( \lambda \):

\[
\Phi = B_{\text{gap}} (b + 2g) \lambda.
\]

Width of backleg is chosen to limit \( B_{\text{yoke}} \) and hence maintain high \( \mu \).
Dipoles can be ‘C core’ ‘H core’ or ‘Window frame’

"C' Core:
Advantages:
  Easy access;
  Classic design;
Disadvantages:
  Pole shims needed;
  Asymmetric (small);
  Less rigid;

The ‘shim’ is a small, additional piece of ferro-magnetic material added on each side of the two poles – it compensates for the finite cut-off of the pole, and is optimised to reduce the 6, 10, 14...... pole error harmonics.
A typical ‘C’ cored Dipole

Cross section of the Diamond storage ring dipole.
H core and window-frame magnets

"Window Frame' Advantages:
- High quality field;
- No pole shim;
- Symmetric & rigid;
Disadvantages:
- Major access problems;
- Insulation thickness

‘H core’:
Advantages:
- Symmetric;
- More rigid;
Disadvantages:
- Still needs shims;
- Access problems.
Providing the conductor is continuous to the steel ‘window frame’ surfaces (impossible because coil must be electrically insulated), and the steel has infinite \( \mu \), this magnet generates perfect dipole field.

Providing current density \( J \) is uniform in conductor:

- \( \mathbf{H} \) is uniform and vertical up outer face of conductor;
- \( \mathbf{H} \) is uniform, vertical and with same value in the middle of the gap;
- \( \rightarrow \) perfect dipole field.
The practical window frame dipole.

Insulation added to coil:

- B increases close to coil insulation surface
- B decreases close to coil insulation surface
- Best compromise
‘Diamond’ storage ring quadrupole.

The yoke support pieces in the horizontal plane need to provide space for beam-lines and are not ferro-magnetic.

Error harmonics include $n = 4$ (octupole) a finite permeability error.
To compensate for the non-infinite pole, shims are added at the pole edges. The area and shape of the shims determine the amplitude of error harmonics which will be present.

The designer optimises the pole by ‘predicting’ the field resulting from a given pole geometry and then adjusting it to give the required quality.

When high fields are present, chamfer angles must be small, and tapering of poles may be necessary.
A first assessment can be made by just examining $B_y(x)$ within the required ‘good field’ region.

Note that the expansion of $B_y(x)_{y=0}$ is a Taylor series:

$$B_y(x) = \sum_{n=1}^{\infty} \{b_n x^{(n-1)}\} = b_1 + b_2x + b_3x^2 + \ldots$$

Also note:

$$\frac{\partial}{\partial x} B_y(x) = b_2 + 2b_3x + \ldots$$

So quad gradient $g \equiv b_2 = \frac{\partial B_y(x)}{\partial x}$ in a quad

But sext. gradient $g_s \equiv b_3 = 2 \frac{\partial^2 B_y(x)}{\partial x^2}$ in a sext.

So coefficients are not equal to differentials for $n = 3$ etc.
Assessing for an adequate design.

A simple judgement of field quality is given by plotting:

- **Dipole:** \[
\frac{B_y(x) - B_y(0)}{B_y(0)} \quad (\Delta B(x)/B(0))
\]
- **Quad:** \[
\frac{dB_y(x)}{dx} \quad (\Delta g(x)/g(0))
\]
- **6poles:** \[
\frac{d^2B_y(x)}{dx^2} \quad (\Delta g_2(x)/g_2(0))
\]

‘Typical’ acceptable variation inside ‘good field’ region:

- \[
\frac{\Delta B(x)}{B(0)} \leq 0.01\%
\]
- \[
\frac{\Delta g(x)}{g(0)} \leq 0.1\%
\]
- \[
\frac{\Delta g_2(x)}{g_2(0)} \leq 1.0\%
\]
Computer codes are now used; eg the Vector Fields codes - ‘OPERA 2D’ and ‘TOSCA’ (3D).

These have:

- finite elements with variable triangular mesh;
- multiple iterations to simulate steel non-linearity;
- extensive pre and post processors;
- compatibility with many platforms and P.C. o.s.

Technique is iterative:

- calculate flux generated by a defined geometry;
- adjust the geometry until required distribution is achieved.
Pre-processor:
The model is set-up in 2D using a GUI (graphics user’s interface) to define ‘regions’:

- steel regions;
- coils (including current density);
- a ‘background’ region which defines the physical extent of the model;
- the symmetry constraints on the boundaries;
- the permeability for the steel (or use the pre-programmed curve);
- mesh is generated and data saved.
Model of Diamond storage ring dipole
Conventional Accelerator Magnets

With mesh added
Pole profile, showing shim and Rogowski side roll-off for Diamond 1.4 T dipole.

‘Close-up’ of pole region.
Note – one eighth of quadrupole could be used with opposite symmetries defined on horizontal and $y = x$ axis.
Data Processor:
either:

• linear - which uses a predefined constant permeability for a single calculation, or

• non-linear - which is iterative with steel permeability set according to B in steel calculated on previous iteration.
Post-processor:

uses pre-processor model for many options for displaying field amplitude and quality:

- field lines;
- graphs;
- contours;
- gradients;
- harmonics (from a Fourier analysis around a pre-defined circle).
Diamond s.r. dipole: \( \frac{\Delta B}{B} = \left\{ \frac{B_y(x) - B(0,0)}{B(0,0)} \right\} \)
typically \( \pm 1 \times 10^4 \) within the ‘good field region’ of \(-12 \text{mm} \leq x \leq +12 \text{mm}\).
2 D Flux density distribution in a dipole
Transverse \((x, y)\) plane in Diamond s.r. dipole;

contours are \(\pm 0.01\%\)

required good field region:
The amplitude and phase of the integrated harmonic components in a magnet provide an assessment:

- when accelerator physicists are calculating beam behaviour in a lattice;
- when designs are judged for suitability;
- when the manufactured magnet is measured;
- to judge acceptability of a manufactured magnet.

Measurement of a magnet after manufacture will be discussed in the section on measurements.
Simpler geometries can be used in some cases. The Diamond dipoles have a Rogawski roll-off at the ends (as well as Rogawski roll-offs at each side of the pole).

See photographs to follow.

This give small negative sextupole field in the ends which will be compensated by adjustments of the strengths in adjacent sextupole magnets – this is possible because each sextupole will have its own individual power supply.
OPERA 3D model of Diamond dipole.
Diamond dipole poles
Diamond
WM quadrupole:
graph is percentage variation in dB/dx vs x at different values of y.

Gradient quality is to be ±0.1 % or better to x = 36 mm.
End chamfering - Diamond ‘W’ quad

Tosca results - different depths 45° end chamfers on $\Delta g/g_0$ integrated through magnet and end fringe field (0.4 m long WM quad).

Thanks to Chris Bailey (DLS) who performed this working using OPERA 3D.
Diamond quadrupoles have an angular cut at the end; depth and angle were adjusted using 3D codes to give optimum integrated gradient.
It is not usually necessary to chamfer sextupole ends (in a d.c. magnet). Diamond sextupole end:
The object of this short section is to:

• introduce the function of the complex variable;
• examine harmonic symmetries using complex notation;
• set the scene for ‘conformal transformations’ (to follow).
Written as

\[ f(z); \]

and

\[ f(z) = u(x,y) + i v(x,y); \]

\[ u (x,y) = \text{Re} f(z); \]

\[ v (x,y) = \text{Im} f(z); \]

For example

\[ f(z) = z^2 - 2z; \]

then:

\[ f(z) = x^2 - y^2 + 2 ixy - 2x -2 iy; \]

so:

\[ u(x,y) = x^2 - y^2 - 2x; \]

\[ v(x,y) = 2(xy - y); \]

We shall be considering functions \( f(z) \) which are
‘continuous’, regular’ and ‘differentiable’.

This places severe restrictions on the class of the two
functions \( u(x,y) \) and \( v(x,y) \).
The differential of a function $f(z)$ is given by:

$$\operatorname{Lt}_{z \to z_0} \left\{ \frac{(f(z) - f(z_0))/(z - z_0)} \right\};$$

To be ‘regular’ (‘differentiable’) the value must not depend on the direction $z$ approaches $z_0$:

- can be along the real axis or the imaginary axis!

This results in the Cauchy-Riemann equations for $u$ and $v$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$  

For the example above: $f(z) = z^2 - 2z$,  

$$\frac{\partial u}{\partial x} = 2x - 2; \quad \frac{\partial v}{\partial y} = 2x - 2;$$  
$$\frac{\partial v}{\partial x} = 2y; \quad \frac{\partial u}{\partial y} = -2y.$$
Cauchy-Riemann and Laplace.

Functions that satisfy Cauchy-Riemann equations also satisfy Laplace's equation in two dimensions:

For $u$: \[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};
\]
differentiate w.r.t. $x$: \[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y};
\]
but: \[
\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]
so: \[
\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2};
\]
and \[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \text{(Laplace's equation in 2D)}.
\]

Similarly for $v$: \[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x};
\]
differentiate w.r.t. $y$: \[
\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y};
\]
\[
= -\frac{\partial^2 v}{\partial x^2}
\]
so: \[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0; \quad \text{(Laplace's equation in 2D)}
\]
Application to potentials

Note that from Cauchy-Riemann:

\[(\frac{\partial u}{\partial x})(\frac{\partial v}{\partial x}) + (\frac{\partial v}{\partial y})(\frac{\partial u}{\partial y}) = 0;\]

So the lines \(u = \text{constant}; v = \text{constant},\) are orthogonal;

In 2D ‘Laplace’ problems (electro-statics, magneto-statics, fluid flow), \(u\) and \(v\) can represent the potential function and the flow function (stream-lines).

So, we can set up a complex potential with real and imaginary parts:
- the \(z\) component of the vector potential;
- the scalar potential.

Note – it does not matter which we make the real and imaginary parts, as long as we are consistent.
We define the complex potential $\Phi$:

$$\Phi = A_z + i\phi;$$

the complex flux density:

$$B = B_x + iB_y$$

Now:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial A_z}{\partial x} + i\frac{\partial \phi}{\partial x};$$

$$= -B_y - iB_x;$$

$$\frac{\partial \Phi}{(i \partial y)} = \frac{1}{i} \frac{\partial A_z}{\partial y} + \frac{\partial \phi}{\partial y};$$

$$= -iB_x - B_y;$$

So we can write:  

$$B^* = i \frac{d \Phi}{dz}$$
The harmonic expansion now becomes:

\[ \Phi = \sum_{n=1}^{\infty} J_n z^n; \]

where \( J_n \) are complex coefficients giving both upright and skew component amplitudes;

For a symmetric 2N pole magnet:

\[ J_n \exp(i\pi n/N) = -J_n \quad \text{all } n; \]

so:

\[ J_n = 0 \quad \text{unless } n = N(2m +1), \ m = 0,1,2,.. \]

<table>
<thead>
<tr>
<th>Magnet</th>
<th>N</th>
<th>Allowed n</th>
</tr>
</thead>
<tbody>
<tr>
<td>dipole</td>
<td>1</td>
<td>1, 3, 5,....</td>
</tr>
<tr>
<td>quadrupole</td>
<td>2</td>
<td>2, 6, 10,...</td>
</tr>
<tr>
<td>sextupole</td>
<td>3</td>
<td>3, 9, 15,....</td>
</tr>
</tbody>
</table>